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Application of Homotopy Perturbation Method for Fractional Partial Differential Equations

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ABSTRACT

Fractional partial differential equations arise from many fields of physics and apply a very important role in various branches of science and engineering. Finding accurate and efficient methods for solving partial differential equations of fractional order has become an active research undertaking. In the present paper, the homotopy perturbation method proposed by J-H He has been used to obtain the solution of some fractional partial differential equations with variable coefficients. Exact and/or approximate analytical solutions of these equations are obtained.

Keywords: Homotopy perturbation method, fractional calculus**,** fractional partial differential equations, Caputo fractional derivative.

1. INTRODUCTION

Fractional differential equations play an important role in fractional calculus. Recently, many authors studied the differential equations of fractional order, see Kilbas *et al.* (2006), Lakshmikantham *et al.* (2009), Miller *et al.* (1993), Hilfert (2000) and Oldham *et al.* (1974). This is due to that the fractional calculus provides an effective and perfect tool for description most of dynamical phenomena in engineering and scientific areas such as, physics, chemical, biology, electrochemistry, electromagnetic, control, porous media and many more, for more details see Diethelm *et al.*(1999), Podlubny (1999) and Metzler *et al.*(1995).

In recent years partial differential equations of fractional order have been received considerable interest and have applied for many problems which are modeled in various areas for instance number of physical phenomena are presented by such equations Mohyud-Din *et al.*(2009), Mohyud-Din and Noor (2008), Usman and Yildirim (2010), Rossikhin and Shitikove (1997) . There are several methods for solving linear and nonlinear fractional partial differential equations for example, homotopy perturbation method Mohyud-Din *et al.* (2011), Momani and Odibat (2007*a*), Odibat (2006), Xu *et al.* (2009), Odibat and Momani (2006), Momani and Odibat (2007), Zhang and He (2006), He (2003), He (1999), He (2000), Abdulaziz *et al.* (2008), He (2004), He (2005), He (2006) homotopy analysis method Jafari and Seifi (2009), Jafari and Seifi (2009*a*), Jafari and Momani (2007) , and variational iteration method Momani and Odibat, (2007*a*).

In the present paper, fractional partial differential equations are obtained from the corresponding integer order equations by replacing the first order or second order time derivatives by a fractional in the Caputo sense of order α with $0< \alpha \leq 1$, $1 < \alpha \leq 2$.

In this work, the homotopy perturbation method proposed by He (1999, 2000) to obtain an approximate solution for partial differential equations of fractional order. Odibat and Momani (2006), applied modification of He's homotopy perturbation method to solve Quadratic Ricccati differential equation of fractional order, also in Momani and Odibat (2007). Momani and Odibat using this method for solving nonlinear partial differential equations of fractional order. Our aim is apply HPM to solve some fractional partial differential equations. The modified homotopy perturbation method Odibat and Momani (2006) shall be adopted.

2. PRELIMINARIES

In this section, we give some basic definitions and properties of fractional calculus theory which used in this paper.

Definition 1

A real function $f(t), t>0$ is said to be in space $C_\nu, \nu \in R$ if there exist a real number $p>\nu$, such that $f(t)=t^q f_1(t)$ where $f(t)\in C(0,\infty)$, and it is said to be in the space C_ν^n if $f^n \in R_\nu$, $n\in N$.

Definition 2

The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_\nu, \nu \geq -1$ is defined as

$$J^\alpha = \frac{1}{\Gamma(\alpha)}\int_0^t f(\tau)(t-\tau)^{\alpha-1}d\tau,\ \alpha>0,\ \tau>0. \tag{1}$$

In the special case $\alpha=0$ we have $J^0 f(t)=f(t)$

For $\beta \geq 0$ and $\gamma \geq -1$, some properties of the operator J^α

(a) $J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t)$
(b) $J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t)$
(c) $J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t)$
(d) $J^\alpha t^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$

Definition 3

The Caputo fractional derivative of $f \in C_{-1}^m$, $m\in N$ is defined as:

$$D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t f^m(\tau)(x-\tau)^{m-\alpha-1}d\tau,\ m-1<\alpha\leq m \tag{2}$$

Lemma 1

If $m-1<\alpha\leq m$, $m\in N$, $f \in C_\nu^m$, $\nu>-1$, then the following two properties hold

(a) $D^\alpha\left[J^\alpha f(t)\right]=f(t)$

(b) $J^{\alpha}\left[D^{\alpha}f(t)\right]=f(t)-\sum_{k=1}^{m-1}f^{(k)}(0)\frac{t^k}{k!}$

3. HOMOTOPY PERTURBATIONMETHOD

The homotopy perturbation method method was proposed in (1999) by Chinese Mathematician He He (1999, 2000, 2003,) is applied to various problems He (2004, 2005, 2006).

$$D^{\alpha}y(t)+L(y(t))+Ny(t)=f(x),\ x>0,\ m-1<\alpha\le m \qquad (3)$$

with boundary conditions

$$B(y,\frac{\partial y}{\partial n})=0,\quad x\in\Gamma \qquad (4)$$

subject to initial condition

$$y^{(k)}(0)=c_k \qquad (5)$$

where $D^{\alpha}y(t)$ is the Caputo fractional derivative of order α, L is a linear operator which might include other fractional derivative operators and N is the nonlinear operator which also might include other fractional derivative of order less than α, B is a boundary operator, $f(x)$ is a known analytic function, Γ is the boundary of the domain Ω.

In view of the homotopy perturbation method, we construct the following homotopy

$$(1-p)D^{\alpha}y(x,p)+p\left[D^{\alpha}y(x,p)+Ly(t,x)+Ny(t,x)-f(x)\right]=0 \qquad (6)$$

or

$$D^{\alpha}y(x,p)+p\left[Ly(t,x)+Ny(t,x)-f(x)\right]=0 \qquad (7)$$

where $p\in[0,1]$ is an embedding parameter. If $p=0$ equations (6) and (7) become

$$D^{\alpha} y = 0 \tag{8}$$

and when $p = 1$ the equations (6) and (7) turn out to be the fractional partial differential equation.

The basic assumption is that the solution of Equation (6) or Equation (7) can be written as a power series in p

$$y = y_0 + p y_1 + p^2 y_2 + \cdots \tag{9}$$

Substituting (9) in (7) and equating the terms with having identical power of p, we obtain the following series of equations

$$\begin{aligned}
&p^0 : D^{\alpha} y_0 = 0,\\
&p^1 : D^{\alpha} y_1 = -L y_0 - N y_0 + f(t),\\
&p^2 : D^{\alpha} y_2 = -L y_1 - N(y_0, y_1),\\
&p^3 : D^{\alpha} y_3 = -L y_2 - N(y_0, y_1, y_2),\\
&\quad \vdots
\end{aligned}$$

4. MODIFIED HOMOTOPY PERTURBATION METHOD

The homotopy perturbation method under development. Many modified various were appeared in literature, for example in Odibat and Momani (2006), Odibat and Momani modified the homotopy perturbation method to solve nonlinear differential equations of fractional order. This modified reduces the nonlinear fractional differential equations to a set of linear ordinary differential equations.

Hereby we will illustrate the general solution procedure of the method. Consider a nonlinear Equation (3)

In view of the homotopy technique, we can constrict the following homotopy

$$y^{(m)} + L(y) - f(x) = p\left[y^{(m)} - N(y) - D^{\alpha} y \right] \tag{10}$$

or

$$y^{(m)} - f_1(x) = p\left[y^{(m)} - L(y) - N(y) - D^{\alpha} y + f_2(x)\right] \tag{11}$$

where $f_1(x)$ be assigned to the zeroth component y_0, and $f_2(x)$ be combined with y_1 *i.e* $f(t,x) = f_1(x) + f_2(x).$

In the case $p = 0$ in Equation (11) we get

$$y^{(m)} = f_1(x) \tag{12}$$

and in the case $p = 1,$ the Equation (11) turn out to be the original fractional differential equation.

Substituting Equation (9) into (11) equating the terms with having identical power of $p,$ we obtain the following series of equations

$$\begin{aligned}
p^0 &: \frac{\partial^m y_0}{\partial x^m} = f_1(t,x), \quad \frac{\partial^k}{\partial x^k} y_0(t,0) = f_k(t),\ k = 0,1,2,\cdots \\
p^1 &: \frac{\partial^m y_1}{\partial x^m} = \frac{\partial^m y_0}{\partial x^m} - Ly_0 - Ny_0 - \frac{\partial^\alpha y_0}{\partial x^\alpha} + f_2(t,x), \\
&\qquad \frac{\partial^k}{\partial x^k} y_1(t,0) = 0, \quad k = 0,1,2,\cdots \\
p^2 &: \frac{\partial^m y_2}{\partial x^m} = \frac{\partial^m y_1}{\partial x^m} - Ly_1 - Ny_1 - \frac{\partial^\alpha y_1}{\partial x^\alpha}, \\
&\qquad \frac{\partial^k}{\partial x^k} y_2(t,0) = 0, \quad k = 0,1,2,\cdots \\
&\vdots
\end{aligned} \tag{13}$$

Note that equation (13) can be solved for y_0, y_1, y_2 by applying J^{α} on both sides and by some computations yields the solution of Equation (3).

5. APPLICATIONS

To incorporate our discussion above, we implement the modified homotopy perturbation method to get the solution of linear and nonlinear partial differential equations of fractional order.

Example 1

In this example we consider the following one dimensional linear inhomogeneous fractional wave equation:

$$D^{\alpha} y(t) + y_t = \frac{x^{(1-\alpha)}}{\Gamma(2-\alpha)} \sin t + x \cos t, \; x > 0, \; 0 < \alpha \le 1 \tag{14}$$

subject to the initial condition

$$y\,(t\,,0) = 0, \tag{15}$$

The exact solution for the special case $\alpha = 1$ is given by

$$y(t, x) = x \sin t \tag{16}$$

First we use standard HPM for solve the equation (14), according to homotopy (7), we have

$$\frac{\partial^{\alpha} y}{\partial x^{\alpha}} = p\left(-y_t + \frac{x^{(1-\alpha)}}{\Gamma(2-\alpha)} \sin t + x \cos t \right) \tag{17}$$

Substituting Equation (9) in to (17) equating the terms with having identical power of p, we obtain the following series of equations

$$\begin{aligned}
&p^0 : \frac{\partial^{\alpha} y_0}{\partial x^{\alpha}} = 0, \quad y_0(x,t) = 0, \\
&p^1 : \frac{\partial^{\alpha} y_1}{\partial x^{\alpha}} = -\frac{\partial y_0}{\partial t} + \frac{x^{(1-\alpha)}}{\Gamma(2-\alpha)} \sin t + x \cos t, \\
&p^2 : \frac{\partial^{\alpha} y_2}{\partial x^{\alpha}} = \frac{\partial y_1}{\partial t}, \\
&\;\;\vdots
\end{aligned} \tag{18}$$

Now applying the operator J^{α} to both sides of Equation (18) and using initial condition yields

$$\begin{aligned} y_0 &= 0, \\ y_1 &= x\sin t + \frac{x^{(\alpha+1)}}{\Gamma(\alpha+2)}\cos t, \\ y_2 &= -\frac{x^{(\alpha+1)}}{\Gamma(\alpha+2)}\cos t + \frac{x^{(2\alpha+1)}}{\Gamma(2\alpha+2)}\sin t, \\ &\vdots \end{aligned} \tag{19}$$

Hence the series of the solution is

$$\begin{aligned} y(t,x) &= y_0(t,x) + y_1(t,x) + y_2(t,x) + \cdots \\ &= x\sin t + \frac{x^{(\alpha+1)}}{\Gamma(\alpha+2)}\cos t - \frac{x^{(\alpha+1)}}{\Gamma(\alpha+2)}\cos t \\ &+ \frac{x^{(2\alpha+1)}}{\Gamma(2\alpha+2)}\sin t + \cdots \end{aligned} \tag{20}$$

By canceling the noise terms in the above series we get

$$y(t,x) = x\sin t$$

which is the exact solution (16) of Equation (14).

Now, we solve Equation (14) by (MHPM). According to homotopy (11), substituting the initial condition (15) into (13), we obtain the following set of partial differential equations

$$\begin{aligned} \frac{\partial y_0}{\partial x} &= \frac{x^{(1-\alpha)}}{\Gamma(2-\alpha)}\sin t + x\cos t, \qquad & y_0(t,0) = 0, \\ \frac{\partial y_1}{\partial x} &= \frac{\partial y_0}{\partial x} - \frac{\partial y_0}{\partial t} - \frac{\partial^{\alpha} y_0}{\partial x^{\alpha}}, \qquad & y_1(t,0) = 0, \\ \frac{\partial y_2}{\partial x} &= \frac{\partial y_1}{\partial x} - \frac{\partial y_1}{\partial t} - \frac{\partial^{\alpha} y_1}{\partial x^{\alpha}}, \qquad & y_2(t,0) = 0, \\ &\vdots \end{aligned} \tag{21}$$

Consequently, solving the above equations for y_0, y_1, y_2 the first few components of homotopy perturbation solution for Equation (14) are derived as follows

$$y_0 = \frac{x^{(2-\alpha)}}{\Gamma(3-\alpha)}\sin t + \frac{x^2}{2}\cos t,$$
$$y_1 = \frac{x^{(2-\alpha)}}{\Gamma(3-\alpha)}\sin t + \frac{x^2}{2}\cos t - \frac{x^{(3-\alpha)}}{\Gamma(4-\alpha)}\cos t + \frac{x^3}{6}\cos t - \frac{x^{(3-2\alpha)}}{\Gamma(4-2\alpha)}\sin t \ ,$$
$$\vdots$$

Hence the series of the solution is

$$\begin{aligned} y(t,x) &= y_0(t,x) + y_1(t,x) + y_2(t,x) + \cdots \\ &= \frac{2x^{(2-\alpha)}}{\Gamma(3-\alpha)}\sin t + x^2\cos t - \frac{2x^{(3-\alpha)}}{\Gamma(4-\alpha)}\cos t + \frac{x^3}{6}\cos t - \frac{x^{(3-2\alpha)}}{\Gamma(4-2\alpha)}\sin t \end{aligned}$$

Hence, if we eliminating the noise terms in the above series of solution the result is the exact solution of (14) when $\alpha = 1$ which is

$$y(t,x) = x\sin t$$

Example 2

We consider the following one differential fractional Burgers equation

$$\frac{\partial^\alpha y}{\partial x^\alpha} + \frac{\partial y}{\partial t} - \frac{\partial^2 y}{\partial t^2} = \frac{2x^{2-\alpha}}{\Gamma(3-\alpha)} + 2t - 2, \quad t \in R,\ 0 < \alpha \le 1, \tag{22}$$

subject to initial condition

$$y(t,0) = t^2 \tag{23}$$

The exact solution for the special case $\alpha = 1$ is given by:

$$y(t,x) = t^2 + x^2 \tag{24}$$

First we use standard HPM for solve the Equation (22), according to homotopy (7), we have

$$\frac{\partial^\alpha y}{\partial x^\alpha} = p\left(-y_t + y_{tt} + \frac{2x^{(2-\alpha)}}{\Gamma(3-\alpha)} + 2t - 2\right) \tag{25}$$

Substituting Equation (9) in to (25) equating the terms with having identical power of p, we obtain the following series of equations

$$\begin{aligned} p^0 &: \frac{\partial^\alpha y_0}{\partial x^\alpha} = 0, \quad y_0(x,t) = t^2, \\ p^1 &: \frac{\partial^\alpha y_1}{\partial x^\alpha} = -\frac{\partial y_0}{\partial t} + \frac{\partial^2 y_0}{\partial t^2} + \frac{2x^{(2-\alpha)}}{\Gamma(3-\alpha)} + 2t - 2, \\ p^2 &: \frac{\partial^\alpha y_2}{\partial x^\alpha} = \frac{\partial y_1}{\partial t} + \frac{\partial^2 y_1}{\partial t^2}, \\ &\vdots \end{aligned} \tag{26}$$

Now applying the operator J^α to both sides of equation (26) and using initial condition yields

$$\begin{aligned} y_0 &= 0, \\ y_1 &= x\sin t + \frac{x^{(\alpha+1)}}{\Gamma(\alpha+2)}\cos t, \\ y_2 &= -\frac{x^{(\alpha+1)}}{\Gamma(\alpha+2)}\cos t + \frac{x^{(2\alpha+1)}}{\Gamma(2\alpha+2)}\sin t, \\ &\vdots \end{aligned} \tag{27}$$

Hence the series of the solution is

$$\begin{aligned} y(t,x) &= y_0(t,x) + y_1(t,x) + y_2(t,x) + \cdots \\ &= x^2 + t^2 \end{aligned} \tag{28}$$

which is the exactly the exact.

Now, we solve Equation (22) by (MHPM). According to homotopy (11), substituting the initial condition (23) into (13), we obtain the following set of partial differential equations.

$$
\begin{aligned}
&\frac{\partial y_0}{\partial x} = 0, \qquad y_0(t,0) = t^2, \\
&\frac{\partial y_1}{\partial x} = \frac{\partial y_0}{\partial x} - \frac{\partial y_0}{\partial t} + \frac{\partial^2 y_0}{\partial t^2} - \frac{\partial^\alpha y_0}{\partial x^\alpha} + \frac{2x^{(2-\alpha)}}{\Gamma(3-\alpha)} + 2t - 2, \quad y_1(t,0) = 0, \\
&\frac{\partial y_2}{\partial x} = \frac{\partial y_1}{\partial x} - \frac{\partial y_1}{\partial t} + \frac{\partial^2 y_1}{\partial t^2} - \frac{\partial^\alpha y_1}{\partial x^\alpha}, \quad y_2(t,0) = 0, \\
&\quad \vdots
\end{aligned}
\tag{29}
$$

Consequently, solving the above equations for y_0, y_1, y_2 the first few components of homotopy perturbation solution for Equation (22) are derived as follows

$$
\begin{aligned}
y_0 &= t^2, \\
y_1 &= \frac{2x^{(3-\alpha)}}{\Gamma(4-\alpha)}, \\
y_2 &= \frac{-2x^{(4-2\alpha)}}{\Gamma(5-2\alpha)} + \frac{2x^{(3-\alpha)}}{\Gamma(4-\alpha)}, \\
&\vdots
\end{aligned}
$$

The results $y_i(t,x) = 0,\ i \geq 3$ in the case of $\alpha = 1$. Hence the series of the solution is

$$
\begin{aligned}
y(t,x) &= y_0(t,x) + y_1(t,x) + y_2(t,x) + \cdots \\
&= x^2 + t^2
\end{aligned}
\tag{30}
$$

which is the exactly the exact.

Example 3

In this we consider Singular linear vibration equation of fractional order:

$$
\frac{\partial^\alpha y}{\partial x^\alpha} = c^2 \left(\frac{\partial^2 y}{\partial t^2} + \frac{1}{t}\frac{\partial y}{\partial t} \right), \qquad x > 0, 1 < \alpha \leq 2,
\tag{31}
$$

subject to initial conditions

$$y(t,0)=t^2, \qquad y_x(t,x)=\frac{c}{t}. \tag{32}$$

with $y(t,x)$ representing the space-time displacement, and c is the wave speed of free vibration.

First we use standard HPM for solve the Equation (31), according to homotopy (7), we have

$$\frac{\partial^\alpha y}{\partial x^\alpha}=pc^2\left(\frac{\partial^2 y}{\partial t^2}+\frac{1}{t}\frac{\partial y}{\partial t}\right) \tag{33}$$

Substituting Equation (9) in to (33) equating the terms with having identical power of p, we obtain the following series of equations

$$\begin{aligned}
&p^0: \frac{\partial^\alpha y_0}{\partial x^\alpha}=0, \;\; y_0(t,x)=t^2, \qquad y_{0x}(t,x)=\frac{c}{t},\\
&p^1: \frac{\partial^\alpha y_1}{\partial x^\alpha}=c^2\left(\frac{\partial^2 y_0}{\partial t^2}+\frac{1}{t}\frac{\partial y_0}{\partial t}\right), \;\; y_0(t,x)=0, \;\; y_{1x}(t,x)=0,\\
&p^2: \frac{\partial^\alpha y_2}{\partial x^\alpha}=c^2\left(\frac{\partial^2 y_1}{\partial t^2}+\frac{1}{t}\frac{\partial y_1}{\partial t}\right), \;\; y_2(t,x)=0, \;\; y_{2x}(t,x)=0,\\
&\quad\vdots
\end{aligned} \tag{34}$$

Now applying the operator J^α to both sides of Equation (34) and using initial condition yields

$$\begin{aligned}
y_0&=t^2+\frac{c}{t}x,\\
y_1&=\frac{4c^2x^\alpha}{\Gamma(\alpha+1)}+\frac{c^3x^{\alpha+1}}{t^3\Gamma(\alpha+2)},\\
y_2&=\frac{-3c^5x^{2\alpha+1}}{t^5\Gamma(2\alpha+2)},\\
&\;\;\vdots
\end{aligned} \tag{35}$$

Hence the series of the solution is

$$\begin{aligned} y(t,x) &= y_0(t,x) + y_1(t,x) + y_2(t,x) + \cdots \\ &= t^2 + \frac{c}{t}x + \frac{4c^2x^\alpha}{\Gamma(\alpha+1)} + \frac{c^3x^{\alpha+1}}{t^3\Gamma(\alpha+2)} - \frac{3c^5x^{2\alpha+1}}{t^5\Gamma(2\alpha+2)} + \cdots \end{aligned} \tag{36}$$

Now, we solve Equation (31) by (MHPM). According to homotopy (11), substituting the initial condition (32) into (13), we obtain the following set of partial differential equations

$$\begin{aligned} &\frac{\partial^2 y_0}{\partial x^2} = 0, \qquad y_0(t,0) = t^2,\ y_{0x}(t,x) = \frac{c}{t} \\ &\frac{\partial^2 y_1}{\partial x} = \frac{\partial^2 y_0}{\partial x^2} + c^2\left(\frac{\partial^2 y_0}{\partial t^2} + \frac{1}{t}\frac{\partial y_0}{\partial t}\right) - \frac{\partial^\alpha y_0}{\partial x^\alpha}, \quad y_1(t,0) = 0,\ y_{1x}(t,x) = 0, \\ &\frac{\partial^2 y_2}{\partial x} = \frac{\partial^2 y_1}{\partial x^2} + c^2\left(\frac{\partial^2 y_1}{\partial t^2} + \frac{1}{t}\frac{\partial y_1}{\partial t}\right) - \frac{\partial^\alpha y_1}{\partial x^\alpha}, \quad y_2(t,0) = 0,\ y_{2x}(t,x) = 0, \\ &\quad\vdots \end{aligned}$$

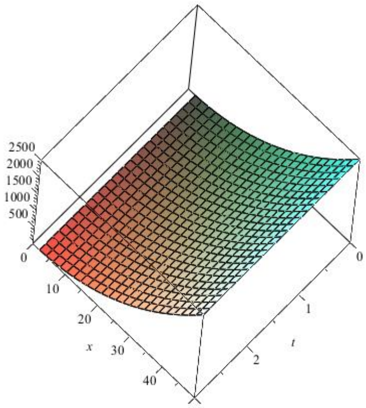
Consequently, solving the above equations for y_0, y_1, y_2 the first few components of homotopy perturbation solution for Equation (31) are derived as follows

$$\begin{aligned} y_0 &= t^2 + \frac{c}{t}x, \\ y_1 &= 2c^2x^2 + \frac{c^3x^3}{6t^3} - \frac{t^2x^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{cx^{3-\alpha}}{t\Gamma(4-\alpha)}, \\ &\ \vdots \end{aligned}$$

Therefore the approximate solution is given by

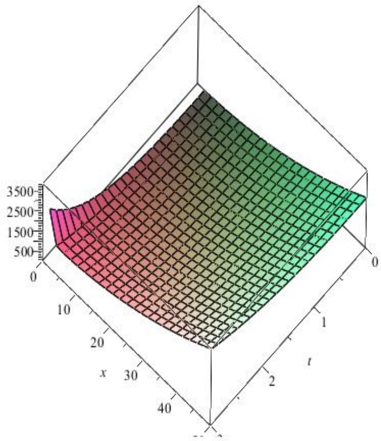
$$\begin{aligned} y(t,x) &= y_0(t,x) + y_1(t,x) + \cdots \\ &= t^2 + \frac{c}{t}x + 2c^2x^2 + \frac{c^3x^3}{6t^3} - \frac{t^2x^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{cx^{3-\alpha}}{t\Gamma(4-\alpha)} \cdots \end{aligned} \tag{37}$$

which is the exact obtained in Abbasbandy (2009) when $\alpha = 2$.



The surface shows the proximate solution $y_{10}(t,x)$ for Equation (31) with respect t and x for c=5

(a)



The surface shows the proximate solution $y_2(t,x)$ for Equation (31) with respect to t and x for $c=5$

(b)

Figure 1: The approximate solution when $\alpha = 2$ and $c = 5$ (a) $y_{10}(t,x)$, (b) $y_2(t,x)$ respectively with respect to t and x

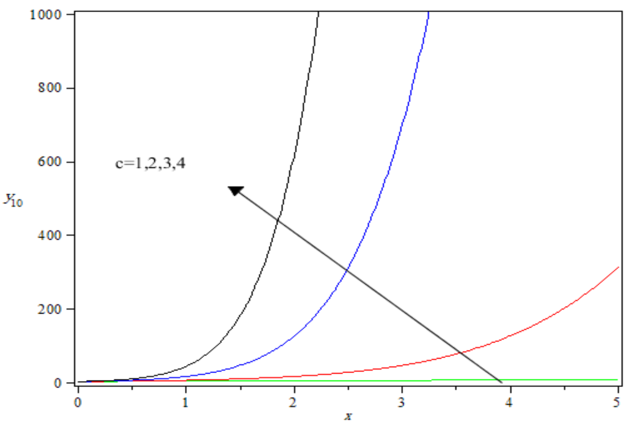


Figure 2: $y_{10}(t,x)$ with respect to $t = 4$ and different c's

The surface of approximate solution has been plotted in Figure 1 ((a) $y_{10}(t,x)$ and (b) $y_2(t,x)$ respectively), we can see that, the displacement increase with increase of t and x. Also, it is increase by calculated more terms of approximate series solution. On the other hand, the Figure 2 shows that the displacement increase fast with the increase x and c with $t=4$.

Example 4

In this example we consider nonlinear time-fractional advection partial differential equation

$$\frac{\partial^\alpha y}{\partial x^\alpha} = y\frac{\partial y}{\partial t} = 2x + t + x^3 + tx^2 \quad x > 0,\ 0 < \alpha \le 1, \tag{38}$$

Subject to initial condition

$$y(t,0) = 0 \tag{39}$$

The exact solution for the special case $\alpha = 1$ is given by

$$y(t,x) = x^2 + tx \tag{40}$$

First we use standard HPM for solve the Equation (38), according to homotopy (7), we have

$$\frac{\partial^\alpha y}{\partial x^\alpha} = p\left(-y_t y + 2x + t + x^3 + tx^2\right) \tag{41}$$

Substituting Equation (9) in to (41) equating the terms with having identical power of p, we obtain the following series of equations

$$\begin{aligned}
&p^0 : \frac{\partial^\alpha y_0}{\partial x^\alpha} = 0, \quad y_0(x,t) = 0,\\
&p^1 : \frac{\partial^\alpha y_1}{\partial x^\alpha} = -y_0\frac{\partial y_0}{\partial t} + 2x + t + x^3 + tx^2,\\
&p^2 : \frac{\partial^\alpha y_2}{\partial x^\alpha} = -y_0\frac{\partial y_1}{\partial t} - y_1\frac{\partial y_0}{\partial t},\\
&\quad \vdots
\end{aligned} \tag{42}$$

Now applying the operator J^{α} to both sides of Equation (42) and using initial condition yields

$$
\begin{aligned}
y_0 &= \frac{x^{\alpha}}{\Gamma(\alpha+1)}, \\
y_1 &= \frac{2x^{(\alpha+1)}}{\Gamma(\alpha+2)} + \frac{tx^{\alpha}}{\Gamma(\alpha+1)} + \frac{x^{(\alpha+3)}}{\Gamma(\alpha+4)} + \frac{2x^{(\alpha+2)}}{\Gamma(\alpha+3)}, \\
y_2 &= -\frac{x^{3\alpha}\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} - \frac{x^{(3\alpha+2)}\Gamma(2\alpha+3)}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)} \\
&\vdots
\end{aligned}
\tag{43}
$$

Hence the series of the solution is

$$
\begin{aligned}
y(t,x) &= y_0(t,x) + y_1(t,x) + y_2(t,x) + \cdots \\
&= \frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{2x^{(\alpha+1)}}{\Gamma(\alpha+2)} + \frac{tx^{\alpha}}{\Gamma(\alpha+1)} + \frac{x^{(\alpha+3)}}{\Gamma(\alpha+4)} + \frac{2x^{(\alpha+2)}}{\Gamma(\alpha+3)} \\
&\quad - \frac{x^{3\alpha}\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} - \frac{x^{(3\alpha+2)}\Gamma(2\alpha+3)}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)}
\end{aligned}
$$

Now, we solve Equation (38) by (MHPM). According to homotopy (11), substituting the initial condition (39) into (13), we obtain the following set of partial differential equations

$$
\begin{aligned}
&\frac{\partial y_0}{\partial x} = 2x + t + x^3 + tx^2, && y_0(t,0) = 0, \\
&\frac{\partial y_1}{\partial x} = \frac{\partial y_0}{\partial x} - y_0\frac{\partial y_0}{\partial t} - \frac{\partial^{\alpha} y_0}{\partial x^{\alpha}}, && y_1(t,0) = 0, \\
&\frac{\partial y_2}{\partial x} = \frac{\partial y_1}{\partial x} - y_0\frac{\partial y_1}{\partial t} - y_1\frac{\partial y_0}{\partial t} - \frac{\partial^{\alpha} y_1}{\partial x^{\alpha}}, && y_2(t,0) = 0, \\
&\vdots
\end{aligned}
\tag{44}
$$

Consequently, solving the above equations for y_0, y_1, y_2 the first few components of homotopy perturbation solution for Equation (38) are derived as follows

$$y_0(t,x)=x^2+tx+\frac{x^4}{4}+\frac{tx^3}{3},$$

$$y_1(t,x)=x^2+tx+\frac{7x^6}{72}+\frac{x^8}{96}-\frac{2x^{(3-\alpha)}}{\Gamma(4-\alpha)}-\frac{tx^{(2-\alpha)}}{\Gamma(3-\alpha)}-\frac{6x^{(5-\alpha)}}{\Gamma(6-\alpha)}-\frac{tx^{(4-\alpha)}}{\Gamma(5-\alpha)},$$

$$\begin{aligned}y_2(t,x)=x^2-\frac{x^4}{4}-\frac{14x^6}{72}-\frac{143tx^8}{2880}+\frac{2783x^{10}}{302400}+\frac{5x^{12}}{8064}-\frac{3x^{(\alpha-3)}}{\Gamma(4-\alpha)}-x^{5-\alpha}\\
\left(-\frac{12}{\Gamma(6-\alpha)}+\frac{3}{(5-\alpha)\Gamma(4-\alpha)}\right)+\frac{x^{(7-\alpha)}}{\Gamma(7-\alpha)}\left(\frac{1}{\Gamma(3-\alpha)}+\frac{2}{3\Gamma(4-\alpha)}\right.\\
\left.\frac{7}{72\Gamma(7-\alpha)}+\frac{2}{\Gamma(5-\alpha)}\right)+x^{9-\alpha}\left(\frac{2}{\Gamma(6-\alpha)}+\frac{1}{\Gamma(5-\alpha)}+\frac{1}{96\Gamma(9-\alpha)}\right)\\
+\frac{2x^{(4-2\alpha)}}{\Gamma(5-2\alpha)}+\frac{6x}{\Gamma(7-2\alpha)}+t\left[x+\frac{2x^3}{3}-\frac{4x^5}{15}-\frac{38x^7}{28835}-\frac{2x^{11}}{2079}-\frac{2x^{(2-\alpha)}}{\Gamma(3-\alpha)}\right.\\
+2x^{(4-\alpha)}\left(\frac{1}{\Gamma(5-\alpha)}+\frac{1}{(4-\alpha)\Gamma(3-\alpha)}\right)+\frac{2x^{(6-\alpha)}}{\Gamma(6-\alpha)}\left(\frac{1}{\Gamma(5-\alpha)}+\frac{1}{15\Gamma(6-\alpha)}\right)\\
\left.+\frac{x^{(8-\alpha)}}{\Gamma(8-\alpha)}\left(\frac{8}{3\Gamma(5-\alpha)}+\frac{1}{63\Gamma(8-\alpha)}\right)+\frac{x^{(3-2\alpha)}}{\Gamma(4-2\alpha)}\right]\end{aligned}$$

Hence the series of the solution is

$$\begin{aligned}y(t,x)&=y_0(t,x)+y_1(t,x)+y_2(t,x)+\cdots\\
&=3x^2+3tx+\frac{x^4}{4}+\frac{tx^3}{3}-\frac{7x^6}{72}-\frac{2tx^5}{15}-\frac{tx^7}{63}-\frac{x^8}{96}-\frac{2x^{(3-\alpha)}}{\Gamma(4-\alpha)}\\
&-\frac{tx^{(2-\alpha)}}{\Gamma(3-\alpha)}-\frac{6x^{(5-\alpha)}}{\Gamma(6-\alpha)}-\frac{tx^{(4-\alpha)}}{\Gamma(5-\alpha)}-\frac{14x^6}{72}-\frac{143tx^8}{2880}+\frac{2783x^{10}}{302400}\\
&+\frac{5x^{12}}{8064}-\frac{3x^{(\alpha-3)}}{\Gamma(4-\alpha)}-x^{5-\alpha}\left(\frac{-12}{\Gamma(6-\alpha)}+\frac{3}{(5-\alpha)\Gamma(4-\alpha)}\right)\end{aligned}$$

$$+\frac{x^{(7-\alpha)}}{\Gamma(7-\alpha)}\left(\frac{1}{\Gamma(3-\alpha)}+\frac{2}{3\Gamma(4-\alpha)}+\frac{7}{72\Gamma(7-\alpha)}+\frac{2}{\Gamma(5-\alpha)}\right)$$
$$+x^{9-\alpha}\left(\frac{2}{\Gamma(6-\alpha)}+\frac{1}{\Gamma(5-\alpha)}+\frac{1}{96\Gamma(9-\alpha)}\right)+\frac{2x^{(4-2\alpha)}}{\Gamma(5-2\alpha)}+\frac{6x^{(6-2\alpha)}}{\Gamma(7-2\alpha)}$$
$$+t\left[x+\frac{2x^3}{3}-\frac{4x^5}{15}-\frac{38x^7}{28835}-\frac{2x^{11}}{2079}-\frac{2x^{(2-\alpha)}}{\Gamma(3-\alpha)}+2x^{(4-\alpha)}\left(\frac{-1}{\Gamma(5-\alpha)}\right.\right.$$
$$\left.+\frac{1}{(4-\alpha)\Gamma(3-\alpha)}\right)+\frac{2x^{(6-\alpha)}}{\Gamma(6-\alpha)}\left(\frac{1}{\Gamma(5-\alpha)}+\frac{1}{15\Gamma(6-\alpha)}\right)+\frac{x^{(8-\alpha)}}{\Gamma(8-\alpha)}$$
$$\left.\left(\frac{8}{3\Gamma(5-\alpha)}+\frac{1}{63\Gamma(8-\alpha)}\right)+\frac{x^{(3-2\alpha)}}{\Gamma(4-2\alpha)}\right]$$

Note that that if we take $\alpha=1$then the first few components the solution of equation (38) is given

$$y_0(t,x)=x^2+tx+\frac{x^4}{4}+\frac{tx^3}{3},$$
$$y_1(t,x)=-\frac{7x^6}{72}-\frac{2x^5}{15}-\frac{tx^7}{63}-\frac{x^8}{96}-\frac{x^4}{4}-\frac{2tx^3}{3},$$
$$y_2(t,x)=\frac{2x^5}{15}-\frac{7x^6}{12}-+\frac{22tx^7}{315}+\frac{143tx^8}{2880}+\frac{38tx^9}{2835}$$
$$+\frac{2783x^{10}}{302400}+\frac{2x^{11}}{2079}+\frac{5x^{12}}{8064},$$

we can cancel the noise term $-\frac{x^4}{4}-\frac{tx^3}{3}$ between y_0 and y_1 and remaining term of y_0 still satisfy the equation, and so on. Hence we get in special case.

$$x^2+t\,x \tag{45}$$

which is exactly the exact solution.

TABLE 1: Comparison of HPM and MHPM with exact solution of equation (38) for different values of α and $t = 0.2$

x	$\alpha = 0.25$		$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1$		Exact Solution
	HPM	MHPM	HPM	MHPM	HPM	MHPM	HPM	MHPM	
0.1	0.65584	0.05707	0.43738	0.05342	0.24625	0.04573	0.12833	0.03003	0.03000
0.2	0.81658	0.14802	0.63668	0.13656	0.42824	0.11602	0.26672	0.08026	0.08000
0.3	0.97727	0.27175	0.80800	0.24860	0.59233	0.21080	0.40525	0.15083	0.15000
0.4	1.15164	0.42749	0.97218	0.38875	0.74332	0.32958	0.53406	0.24173	0.24000
0.5	1.34334	0.61449	1.13773	0.55624	0.88264	0.47176	0.64323	0.35261	0.35000
0.6	1.55367	0.83174	1.30923	0.74997	1.01071	0.63627	0.72281	0.48249	0.48000
0.7	1.78326	1.07752	1.48946	0.96806	1.12753	0.82120	0.76274	0.62940	0.63000
0.8	2.03243	1.34874	1.68024	1.20728	1.23283	1.02318	0.75281	0.78983	0.80000
0.9	2.30143	1.64033	1.88280	1.46233	1.32609	1.23674	0.68265	0.95815	0.99000
1.0	2.59049	1.94428	2.09795	1.72495	1.40660	1.45343	0.54167	1.12569	1.20000

TABLE 2: Absolute Error

x	$\lVert y_{Exact} - y_{HPM} \rVert$	$\lVert y_{Exact} - y_{MHPM} \rVert$
0.1	9.8E-2	3.1E-5
0.2	1.8E-1	2.6E-4
0.3	2.6E-1	8.3E-4
0.4	2.9E-1	1.7E-3
0.5	2.9E-1	2.6E-3
0.6	2.4E-1	2.4E-3
0.7	1.3E-1	6.1E-3
0.8	4.7E-2	1.0E-2
0.9	3.1E-1	3.2E-2
1.0	6.4E-1	5.6E-2

Table 1 shows the approximate solution for Equation (38) obtained for different values of α using homotopy perturbation method and modified homotopy perturbation method. The value $\alpha = 1$ is the only case for which we know exact solution $y(t,x) = x^2 + tx$ and our approximate solution using modified homotopy perturbation method is more accurate than homotopy perturbation method. It is be noted that the second-order term series and third-order term series was used in evaluating the approximate solutions by modified homotopy perturbation method and homotopy perturbation method respectively for Table 1. On the other hand, Table 2 shows the absolute error of HPM and MHPM series solution.

6. CONCLUSIONS

The modified homotopy perturbation method (MHPM) suggested in this work is an efficient method for calculating analytical approximate solutions for fractional linear and nonlinear partial differential equations. The presented examples illustrate that the method capable reducing the volume of computational work as compared to standard homotopy perturbation method. Further, the agreement between the approximate and the exact solution in all examples shows the efficiency of the method and related phenomena give the method much wider applicability.

REFERENCES

Abbasbandy, S. and Shivanian, E. (2009). Solution of singular linear vibrational BVPs by the homotopy analysis method. *Journal of Numerical Mathematics and Stochastic.* **1**(1): 77-84.

Abdulaziz, O., Hashim, I. and Momani, S. (2008). Application of homotopy-perturbation method to fractional IVPs. *Journal of Computational and Applied Mathematics.* **216**: 574-584.

Diethelm K. and Freed, A. D. (1999). On the solution of nonlinear fractional Order differential equations used in modeling of viscoelasticity, *in Scientific Computing in Chemical Engineering II-Computational Fluid Dynamics, Reaction Engineering and Molecular Properties*, F. Keil, W. Mackens, H. Voss and J. Werther, Eds., PP. 217-224. Springer, Heidelberg, Germany.

He, J. H. (1999). Homotopy perturbation technique. *Computer Methods in Applied Mechanics and Engineering*. **178**: 257-262.

He, J. H. (2000). A coupling method of homotopy technique and perturbation technique for nonlinear problems. *International Journal of Non-Linear Mechanics.* **35**: 37-43.

He, J. H. (2003). Homotopy perturbation method: a new nonlinear analytic Technique. *Applied Mathematics and Computation*. **135**: 73-79.

He, J. H. (2004). The homotopy perturbation method for nonlinear oscillators with discontinuities. *Applied Mathematics and Computation*. **151**: 287-292.

He, J. H. (2005). Application of homotopy perturbation method to nonlinear wave equations. *Chaos Solitons Fractals*. **26**(3): 695-700.

He, J. H. (2006). Homotopy perturbation method for solving boundary value Problems. *Physics Letters A.* **350** (1-2): 87-79.

Hilfert, R. (2000). *Applications of fractional calculus in physics*. USA: World Scientific, River Edge, NJ,.

Jafari, H. and Seifi, S.(2009). Homotopy analysis method for solving linear and nonlinear diffusion-wave equation. *Communications in Nonlinear Science and Numerical Simulation*. **14**: 2006-2012.

Jafari, H. and . Momani, S. (2007). Solving fractional diffusion and wave equations by modified homotopy perturbation method. *Physics Letters A.* **370**: 388–396.

Jafari, H. and Seifi, S. (2009*a*). Solving a system of nonlinear fractional partial differential equations using homotopy analysis method. *Communications in Nonlinear Science and Numerical Simulation.* **14**: 1962-1969.

Kilbas, A. A., Srivastava, H. M. and Trujilo, J. J. (2006). *Theory and Applications of fractional differential equations*. Vol. 204 of North-Holland Mathematics Studies, Elsevier Science B.V., Amesterd The Netherlands.

Lakshmikantham, V., Leela, S. and Devi, J. V. (2009). *Theory of fractional Dynamic Systems*. Cambridge Scientific.

Metzler, K., Schick, W., Kilian, H. G. and Nonnenmacher, T. F. (1995). Relaxation in field polmers: a fractional calculus approach. *Journal of Chemical Physics*. **103**(16) : 7180-7186.

Miller, K. S. and Ross, B. (1993). *An introduction to the fractional calculus and fractional differential equations*. New York, NY, USA: A Wiley-Interscience Publication, John Wiley and Sons.

Mohyud-Din, S. T., Noor, K. I., Noor, M. A. (2009). Traveling wave solutions of seventh-order generalized Kdv equations using He's polynomials. *International Journal of Nonlinear Sciences and Numerical Simulation.* **10**(2): 223-229.

Mohyud-Din, S. T., Yildirim, A. and Usman, M. (2011). Homotopy analysis Method for fractional partial differential equations. *International Journal of the Physical Sciences*. **6**(1): 136-145.

Mohyud-Din, S.T. and Noor, M. A. (2008). Homotopy perturbation method For nonlinear higher-order boundary value problems. *International Journal of Nonlinear Sciences and Numerical Simulation.* **9**(4): 395-408.

Momani, S. and Odibat, Z. (2007). Homotopy perturbation method for nonlinear partial differential equations of fractional order. *Physics Letters A*. **365**: 345–350.

Momani, S. and Odibat, Z. (2007*a*). Comparison between the homotopy perturbation method and the variational iteration method for linear fractional partial differential equations. *Computers & Mathematics with Applications*. **54**: 910-919.

Odibat, Z. (2006). A new modification of the homotopy perturbation Method for linear and nonlinear operators. *Applied Mathematics and Computation*.**189**(1): 746–753.

Odibat, Z. and Momani, S. (2008). Modified homotopy perturbation method: Application to quadratic Riccati differential equation of fractional order. *Chaos, Solitions and Fractals*. **36**: 167–174.

Oldham, K. B. and Spanier, J. (1974). *The fractional calculus*. Mathematics in Science and Engineering V198". Academic Press.

Podlubny, I. (1999). *Fractional differential equation*. Mathematics in Science and Engineering, V198 of Mathematics in Science and Engineering. San Diego, California, USA: Academic Press.

Rossikhin, Y. A. and Shitikove, M. V. (1997). Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids. *Applied Mechanics Reviews*. **50**: 15-67.

Usman, M. S. T. and Yildirim, A. (2010). An iterative algorithm for boundary value problems using Pade approximates. *World Applied Science J*. **10**(6): 637-644.

Xu, H., Liao, S. J and You, X-C. (2009). Analysis of nonlinear fractional partial differential equations with the homotopy analysis method. *Communications in Nonlinear Science and Numerical Simulation*. **14**: 1152-1156.

Zhang, L. N. and He, J. H. (2006). Homotopy perturbation method for the solution of electrostatic potential differential equation. *Mathematical Problems in Engineering*. Article ID 83878.